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1997 J. Phys. A: Math. Gen. 30 L149

(http://iopscience.iop.org/0305-4470/30/7/001)

J. Phys. A: Math. Gen. **30** (1997) L149–L154. Printed in the UK PII: S0305-4470(97)78543-3

**LETTER TO THE EDITOR**

# Quantum Heisenberg–Weyl algebras

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Received 8 October 1996, in final form 30 December 1996

**Abstract.** All Lie bialgebra structures on the Heisenberg–Weyl algebra $[A_-, A_+] = M$ are classified and explicitly quantized. The complete list of quantum Heisenberg–Weyl algebras so obtained includes new multiparameter deformations, most of them being of the non-coboundary type.

A Hopf algebra deformation of a universal enveloping algebra $Ug$ defines in a unique way a Lie bialgebra structure $(g, \delta)$ on $g$ [1]. The cocommutator $\delta$ provides the first-order terms in the deformation of the coproduct, and can be seen as the natural tool to classify quantum algebras. Moreover, this well known statement suggests the relevance of the inverse problem, i.e. to find a method to construct, given an arbitrary Lie bialgebra, a Hopf algebra quantization of it.

This question has been addressed recently in [2], where a very general construction of a deformed coassociative coproduct linked to a given Lie bialgebra $(g, \delta)$ has been presented. Such Lie bialgebra quantization formalism, inspired by the paper [3] (see also [4]), has been shown to be universal for the oscillator algebra: multiparametric coproducts corresponding to all coboundary oscillator Lie bialgebra structures can be obtained in that way (for the oscillator algebra non-coboundary structures do not exist [5]). To complete the structure of quantum algebras, deformed commutation rules can be found by imposing the homomorphism condition for the coproduct (counit and antipode can be also easily derived).

In this letter we show that all Heisenberg–Weyl Lie bialgebras can be completely quantized by making use of this formalism. This result enhances the advantages of such an approach in order to obtain a full chart of Hopf algebra deformations of physically relevant algebras.

Firstly, we find the most general form of all families of Heisenberg–Weyl Lie bialgebras. It is remarkable that, in contrast to the oscillator case, now there exists only one coboundary bialgebra among them. Afterwards, it is shown how all these Lie bialgebras can be classified and 'exponentiated' to get the quantum coproducts by means of the formalism introduced in [2]. We also find all deformed commutation rules, thus obtaining a complete list of quantum deformations of this algebra, whose properties are briefly commented. This exhaustive description is fully complementary with respect to the quantum group results already known either from a Poisson–Lie construction [6] or from an $R$-matrix approach [7].

Let us fix the notation. The Heisenberg–Weyl Lie algebra $h_3$ is generated by $A_+$, $A_-$ and $M$ with Lie brackets

$$[A_-, A_+] = M \qquad [M, \cdot\,] = 0. \tag{1}$$

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A $3 \times 3$ real matrix representation $D$ of (1) is given by

$$D(A_+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad D(A_-) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad D(M) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

The expression for a generic element of the Heisenberg–Weyl group $H_3$ coming from this representation is

$$D(T) = \exp\{m D(M)\} \exp\{a_- D(A_-)\} \exp\{a_+ D(A_+)\} = \begin{pmatrix} 1 & a_- & m + a_- a_+ \\ 0 & 1 & a_+ \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

and the group law for the coordinates $m$, $a_-$ and $a_+$ is obtained by means of matrix multiplication $D(T'') = D(T') \cdot D(T)$:

$$m'' = m + m' - a_- a'_+ \qquad a''_+ = a'_+ + a_+ \qquad a''_- = a'_- + a_-. \tag{4}$$

Heisenberg–Weyl Lie bialgebras $(h_3, \delta)$ will be defined by the cocommutator $\delta : h_3 \to h_3 \otimes h_3$ such that

(i) $\delta$ is a 1-cocycle, i.e.

$$\delta([X, Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)] \qquad \forall X, Y \in h_3 \tag{5}$$

(ii) the dual map $\delta^* : h_3^* \otimes h_3^* \to h_3^*$ is a Lie bracket on $h_3^*$.

From (ii), we consider an arbitrary skewsymmetric cocommutator:

$$\begin{aligned} \delta(A_-) &= a_1 A_- \wedge A_+ + a_2 A_- \wedge M + a_3 A_+ \wedge M \\ \delta(A_+) &= b_1 A_- \wedge A_+ + b_2 A_- \wedge M + b_3 A_+ \wedge M \\ \delta(M) &= c_1 A_- \wedge A_+ + c_2 A_- \wedge M + c_3 A_+ \wedge M \end{aligned} \tag{6}$$

where $a_i, b_i, c_i$ $(i = 1, 2, 3)$ are real parameters. If we impose on (6) the cocycle condition (5) we obtain

$$c_1 = 0 \qquad c_2 = b_1 \qquad c_3 = -a_1. \tag{7}$$

Since the dual $h_3^*$ with generators $\{m, a_-, a_+\}$ must be a Lie algebra, the Jacobi identity on the bracket $\delta^*$ gives rise to two additional conditions:

$$a_1(b_3 - a_2) - 2b_1 a_3 = 0 \qquad b_1(a_2 - b_3) - 2a_1 b_2 = 0. \tag{8}$$

Hence, the most general Heisenberg–Weyl bialgebra has commutation relations (1) and cocommutators

$$\begin{aligned} \delta(A_-) &= a_1 A_- \wedge A_+ + a_2 A_- \wedge M + a_3 A_+ \wedge M \\ \delta(A_+) &= b_1 A_- \wedge A_+ + b_2 A_- \wedge M + b_3 A_+ \wedge M \\ \delta(M) &= b_1 A_- \wedge M - a_1 A_+ \wedge M \end{aligned} \tag{9}$$

with the six parameters $a_i$, $b_i$ verifying (8).

It is also known that the dual Lie bracket $\delta^*$ gives the linear part of the (unique) Poisson–Lie structure on the group linked to $\delta$ [8]. Therefore, starting from the classification of Poisson–Lie Heisenberg groups given in [6] and taking into account the change of local coordinates on the Heisenberg group

$$x_1 = a_- \qquad x_2 = a_+ \qquad x_3 = m + a_- a_+ \tag{10}$$

it is straightforward to prove that the full Poisson–Lie bracket associated to $\delta$ reads

$$\begin{aligned} \{a_-, a_+\} &= a_1 a_- + b_1 a_+ \\ \{a_-, m\} &= a_2 a_- + b_2 a_+ + b_1 m - \tfrac{1}{2} a_1 a_-^2 \\ \{a_+, m\} &= a_3 a_- + b_3 a_+ - a_1 m + \tfrac{1}{2} b_1 a_+^2. \end{aligned} \tag{11}$$

In other words, if (4) is read as a coproduct on $\mathrm{Fun}(H_3)$, it is easy to check that the group law turns out to be a Poisson algebra homomorphism with respect to (11).

Finally, let us find out for which values of the parameters we have coboundary Lie bialgebras (see [9]). So, we investigate the most general skewsymmetric element $r$ of $h_3 \otimes h_3$ such that

$$\delta(X) := [1 \otimes X + X \otimes 1, r] \qquad X \in h_3 \tag{12}$$

defines a Lie bialgebra. This is equivalent to imposing the Schouten bracket $[[r, r]]$ to be a solution of the modified classical Yang–Baxter equation (YBE)

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]\,] = 0 \qquad X \in h_3. \tag{13}$$

Explicitly, we consider three real-valued coefficients $\xi$, $\beta_+$ and $\beta_-$ and write

$$r = \xi A_+ \wedge A_- + \beta_+ A_+ \wedge M + \beta_- A_- \wedge M. \tag{14}$$

The Schouten bracket of this element is given by

$$[[r, r]] = -\xi^2 M \wedge A_+ \wedge A_-. \tag{15}$$

This bracket is found to fulfill automatically the modified classical YBE (13). Therefore, (14) is always a classical $r$-matrix. The cocommutator (12) derived from it reads

$$\delta(A_+) = -\xi A_+ \wedge M \qquad \delta(A_-) = -\xi A_- \wedge M \qquad \delta(M) = 0. \tag{16}$$

Thus, we conclude that there exists only one non-trivial coboundary Heisenberg–Weyl Lie bialgebra which is characterized by

$$a_1 = a_3 = b_1 = b_2 = 0 \qquad a_2 = b_3 = -\xi. \tag{17}$$

The case $\xi = 0$ gives rise to a solution of the classical YBE, but now the cocommutator vanishes.

Let us go back to the four-parameter family of bialgebras given by (8) and (9). It is easy to check that equations (8) have three disjoint types of solutions

Type $\mathrm{I}_+$: $a_1 \neq 0$, $b_2 = -a_3\, b_1^2/a_1^2$, $b_3 = a_2 + 2b_1 a_3/a_1$ and $a_2, a_3, b_1$ arbitrary.

Type $\mathrm{I}_-$: $a_1 = 0$, $b_1 \neq 0$, $a_3 = 0$, $a_2 = b_3$ and $b_2, b_3$ arbitrary.

Type II: $a_1 = 0$, $b_1 = 0$ and $a_2, a_3, b_2, b_3$ arbitrary.

So, we have three (multiparametric) families of Lie bialgebras. To quantize them, we have to check that, within each family [2]:

(a) there exists some set $\{H_i\}$ of commuting generators of $g$ such that $\delta(H_i) = 0$ (these will be the primitive generators after quantization);

(b) for the remaining generators $X_j$, their cocommutator $\delta(X_j)$ must only contain terms of the form $X \wedge H$ (neither $X_l \wedge X_m$ nor $H_n \wedge H_p$ contributions are allowed).

Finally, we have to take into account the fact that two Lie bialgebra structures of a Lie algebra $g$ are equivalent if there exists an automorphism of $g$ that transforms one into the other. As we shall see, some automorphisms of the Heisenberg algebra will help us to get bialgebras fulfilling conditions (a) and (b).

*Type $\mathrm{I}_+$.* This is a family of Lie bialgebras which has, for general values of the parameters, no primitive generator $\delta(H) = 0$. However, if we define

$$A'_+ := A_+ - \frac{b_1}{a_1} A_- + \left( \frac{b_1\, a_3}{a_1^2} + \frac{a_2}{a_1} \right) M \qquad a_1 \neq 0 \tag{18}$$

it is immediate to check that, in this new basis, the type $\mathrm{I}_+$ bialgebras have the following cocommutator:

$$\begin{aligned}&\delta(A_-) = -a_1\, A'_+ \wedge A_- + a_3 A'_+ \wedge M\\ &\delta(A'_+) = 0\\ &\delta(M) = -a_1 A'_+ \wedge M.\end{aligned} \tag{19}$$

The automorphism (18) has shown the parameters $b_1$ and $a_2$ to be superfluous.

The coproduct that quantizes the resultant biparametric family (19) can now be obtained. First, we see that this family of bialgebras verifies conditions (a) and (b) with $A'_+$ being the primitive generator (from now on, we shall write $A_+$ instead of $A'_+$). Following [2] we write the non-vanishing cocommutators in (19) in the matrix form

$$\delta\begin{pmatrix}A_-\\ M\end{pmatrix} = \begin{pmatrix}-a_1 A_+ & a_3 A_+\\ 0 & -a_1 A_+\end{pmatrix} \dot{\wedge} \begin{pmatrix}A_-\\ M\end{pmatrix}. \tag{20}$$

In this way, the coproduct for non-primitive generators will be formally given by

$$\Delta\begin{pmatrix}A_-\\ M\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \dot{\otimes} \begin{pmatrix}A_-\\ M\end{pmatrix} + \sigma\left(\exp\left\{\begin{pmatrix}a_1 A_+ & -a_3 A_+\\ 0 & a_1 A_+\end{pmatrix}\right\} \dot{\otimes} \begin{pmatrix}A_-\\ M\end{pmatrix}\right) \tag{21}$$

where $\sigma$ is the exchange operator on the tensor product. By computing explicitly the exponential, we find that

$$\begin{aligned}&\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1 \qquad \Delta(M) = 1 \otimes M + M \otimes \mathrm{e}^{a_1 A_+}\\ &\Delta(A_-) = 1 \otimes A_- + A_- \otimes \mathrm{e}^{a_1 A_+} - a_3 M \otimes A_+\, \mathrm{e}^{a_1 A_+}.\end{aligned} \tag{22}$$

The next step is the search for deformed commutation rules compatible with (22). They turn out to be

$$[A_-, A_+] = M \qquad [A_-, M] = \tfrac{1}{2} a_1 M^2 \qquad [A_+, M] = 0. \tag{23}$$

Finally, counit and antipode are deduced

$$\epsilon(X) = 0 \qquad X \in \{A_-, A_+, M\} \tag{24}$$

$$\begin{aligned}&\gamma(A_+) = -A_+ \qquad \gamma(M) = -M\, \mathrm{e}^{-a_1 A_+}\\ &\gamma(A_-) = -A_-\, \mathrm{e}^{-a_1 A_+} - a_3 M A_+\, \mathrm{e}^{-a_1 A_+}\end{aligned} \tag{25}$$

and the Hopf algebra $U_{a_1,a_3}(h_3)$ that quantizes the family of (non-coboundary) Heisenberg–Weyl bialgebras (19) is obtained.

It is remarkable that in this quantum deformation the parameter $a_3$ is not involved in the deformed commutation rules. Recall that (23) was firstly obtained in [7] starting from a quantum Heisenberg group and by applying a duality method (coproduct (22) could not be found).

On the other hand, a physically suggestive observation comes from the fact that the generator $M$ is neither central nor primitive (recall the role that the non-primitive mass generator of quantum extended Galilei algebra plays in one-dimensional magnon systems [10]). The central element $\mathcal{C}$ is now

$$\mathcal{C} = M\, \mathrm{e}^{-a_1 A_+/2}. \tag{26}$$

This element labels the following differential realization of $U_{a_1,a_3}(h_3)$:

$$A_+ = x \qquad A_- = \lambda\, \mathrm{e}^{a_1 x/2} \partial_x \qquad M = \lambda\, \mathrm{e}^{a_1 x/2} \tag{27}$$

where $\lambda$ is the eigenvalue of $\mathcal{C}$. Note also that, by introducing $\mathcal{C}$ as a new generator instead of $M$, relations (23) turn into

$$[A_-, A_+] = \mathcal{C}\, \mathrm{e}^{a_1 A_+/2} \qquad [A_-, \mathcal{C}] = 0 \qquad [A_+, \mathcal{C}] = 0. \tag{28}$$

Finally note that, if $\tilde{A}$, $\tilde{A}_+$ and $\tilde{A}_-$ are the generators of the non-standard quantum deformation $U_z sl(2,\mathbb{R})$ [11], the quantum Heisenberg algebra $U_{a_1,0}(h_3)$ can be obtained as the contraction $\varepsilon \to 0$ defined by

$$M = -\varepsilon\tilde{A}, \qquad A_+ = \tilde{A}_+ \qquad A_- = \varepsilon\tilde{A}_- \qquad a_1 = 2z. \tag{29}$$

As could be expected from the non-coboundary character of $U_{a_1,0}(h_3)$, the universal $R$-matrix of $U_z sl(2,\mathbb{R})$ diverges under (29).

*Type I$_-$.* After specializing the corresponding parameters we find a three-parameter cocommutator also with no primitive generators. However, the definition of $A'_-$ by means of the automorphism

$$A'_- := A_- - \frac{b_3}{b_1}M \qquad b_1 \neq 0 \tag{30}$$

implies that this family of Lie bialgebras is given by

$$\begin{aligned}
&\delta(A'_-) = 0\\
&\delta(A_+) = b_1 A'_- \wedge A_+ + b_2 A'_- \wedge M\\
&\delta(M) = b_1 A'_- \wedge M.
\end{aligned} \tag{31}$$

In particular, the parameter $b_3$ has been reabsorbed, and (31) can be quantized. Moreover, these type I$_-$ structures are essentially the same as the type I$_+$ (19), but reversing the role of $A_-$ and $A_+$. Once again, another Heisenberg algebra automorphism given by

$$A_+ \to A_- \qquad A_- \to A_+ \qquad M \to -M \tag{32}$$

would make both types of bialgebras explicitly equivalent. Therefore, we omit the explicit quantization leading to the algebra $U_{b_1,b_2}(h_3)$.

*Type II.* If $a_1$ and $b_1$ vanish, the cocommutator (9) reads

$$\begin{aligned}
&\delta(A_-) = a_2 A_- \wedge M + a_3 A_+ \wedge M\\
&\delta(A_+) = b_2 A_- \wedge M + b_3 A_+ \wedge M\\
&\delta(M) = 0.
\end{aligned} \tag{33}$$

In this case, $M$ is the primitive generator and no extra manipulation is needed in order to quantize this family of bialgebras. We write (33) in matrix form:

$$\delta\begin{pmatrix} A_- \\ A_+ \end{pmatrix} = \begin{pmatrix} -a_2 M & -a_3 M \\ -b_2 M & -b_3 M \end{pmatrix} \dot{\wedge} \begin{pmatrix} A_- \\ A_+ \end{pmatrix}. \tag{34}$$

Hence, the corresponding coproduct is given by

$$\Delta\begin{pmatrix} A_- \\ A_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \dot{\otimes} \begin{pmatrix} A_- \\ A_+ \end{pmatrix} + \sigma\left(\exp\left\{\begin{pmatrix} a_2 M & a_3 M \\ b_2 M & b_3 M \end{pmatrix}\right\} \dot{\otimes} \begin{pmatrix} A_- \\ A_+ \end{pmatrix}\right). \tag{35}$$

Although the four parameters describing this quantum algebra are arbitrary, in order to derive the commutation rules compatible with (35) it will suffice to write

$$E := \exp\left\{\begin{pmatrix} a_2 M & a_3 M \\ b_2 M & b_3 M \end{pmatrix}\right\} = \begin{pmatrix} E_{11}(M) & E_{12}(M) \\ E_{21}(M) & E_{22}(M) \end{pmatrix}. \tag{36}$$

In this way, the explicit quantum coproduct will be

$$\begin{aligned}
&\Delta(M) = 1 \otimes M + M \otimes 1\\
&\Delta(A_-) = 1 \otimes A_- + A_- \otimes E_{11}(M) + A_+ \otimes E_{12}(M)\\
&\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes E_{22}(M) + A_- \otimes E_{21}(M).
\end{aligned} \tag{37}$$

Now, by taking into account the following property,

$$E_{11}(M)\,E_{22}(M) - E_{12}(M)\,E_{21}(M) = \mathrm{e}^{(a_2+b_3)\,M} \qquad (38)$$

it is straightforward to prove that the four-parameter coproduct (37) is an algebra homomorphism with respect to the deformed commutation rules

$$[A_-, A_+] = \frac{\mathrm{e}^{(a_2+b_3)\,M} - 1}{a_2 + b_3} \qquad [A_-, M] = 0 \qquad [A_+, M] = 0. \qquad (39)$$

Due to the preservation of $M$ as central element, counit and antipode are easily deduced. These operations complete the construction of the multiparametric quantum algebra $U_{a_2,a_3,b_2,b_3}(h_3)$. These type II quantizations were studied in [4] with no reference to Lie bialgebra structures.

The well known coboundary quantization is a particular subcase with $a_3 = b_2 = 0$ and $a_2 = b_3 = -\xi$. A universal $R$-matrix (which is not a solution of the quantum YBE) for it was obtained in [12], and a $*$-product quantizing the corresponding Poisson–Lie Heisenberg group was introduced in [9].

AB and FJH are partially supported by DGICYT (Project PB94-1115) from the Ministerio de Educación y Ciencia de España.

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